

ON EMBEDDING 3-MANIFOLDS IN 4-SPACE

PATRICK M. GILMER and CHARLES LIVINGSTON

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INTRODUCTION

THIS PAPER presents new obstructions to embedding 3-manifolds in 4-space as well as some constructions for embedding. It has long been known that not all 3-manifolds embed [9]. On the other hand, every closed 3-manifold embeds in 5-space [10, 17, 19]. Our main contribution lies in considering signature invariants associated to covers of the 3-manifold that must extend over one component of the complement. We focus on rational homology spheres as our results are prettiest in this case. However, the methods apply much more generally.

Section 1 defines the relevant invariants that we use. In §2, we state and prove our main theorem. We consider connected sums of lens space in §3 and then 3-manifolds with framed link descriptions of a certain type in §4. Section 5 applies the theorem to the study of doubly null-concordant knots in S^3 . In §6, we consider manifolds with the homology of $S^1 \times S^2$.

We work in the smooth category. All manifolds will be oriented. Homology will be with integral coefficients unless indicated otherwise.

§1. INVARIANTS OF 3-MANIFOLDS

Let M be a 3-manifold equipped with an almost framing $\mathcal{F}$ (this is a trivialization of the tangent bundle $\mathcal{T}(M - \text{a point})$). We can always find a framed simply connected 4-manifold V with boundary M such that the two framings on $\mathcal{T}(M - \text{point}) \oplus \epsilon$ match [12] (this is originally a theorem of Milnor). One defines $\mu(M, \mathcal{F}) = \text{Sign } V \bmod 16$. See [11] for example.

In this situation, one may also define a quadratic form

$$q_{\mathcal{F}} : \text{Tor } H_1(M) \rightarrow \mathbb{Q}/\mathbb{Z}$$

due to Morgan and Sullivan [16]. In our context, it can be defined as follows. Given an element $x \in \text{Tor } H_1(M)$, represent it by an embedded curve γ in M . We will now use $\mathcal{F}$ to construct a framing for the normal bundle of γ (this framing is only determined mod 2). Since $H_1(V) = 0$, we can find an embedded surface G in V with boundary γ . Pick a framing to normal bundle of G in V and restrict this framing to γ . If we picked a different V and G , we may obtain a different framing for the normal bundle of γ . However, it is easy to see that the framings agree mod 2.

Now let γ' denote a parallel copy of γ obtained by pushing γ off itself with this framing. Now $r\gamma'$ will be nullhomologous in M for some r and we can represent this null homology by an embedded surface F with boundary r parallel copies of γ' very near to γ' . Define $q_{\mathcal{F}}(x, x) = 1/2r(\gamma \cdot F)$ where $\gamma \cdot F$ denotes the algebraic intersection number of γ and F .

The quadratic form satisfies the following relations [16]:

$$q(x + y) = q(x) + q(y) + l(x, y) \tag{1}$$

$$q(\lambda, x) = \lambda^2 q(x). \tag{2}$$

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Here $l(x, y)$ denotes the ordinary linking form. It follows from the above equations that l determines q on the odd torsion of $H_1(M)$. See ([16], p. 497).

In practice the framing for the normal bundle of γ can be obtained in the following curious manner. First isotope $\mathcal{F}$ so that restricted to γ the third vector is tangent to γ , then the first two vectors give a framing to the normal bundle of γ . Now change this framing by an odd twist and use this new framing!

Because M -point has the homotopy type of a 2-complex $H^1(M, \mathbb{Z}_2) = H^1(M\text{-point}, \pi_1(SO(3))) = [M\text{-point}, SO(3)]$. Thus $H^1(M, \mathbb{Z}_2)$ acts freely and transitively on the set of isotopy classes of almost framings of M . We let $\psi\mathcal{F}$ denote the action of ψ on $\mathcal{F}$.

We now recall the definition of the Casson–Gordon invariant of a finite cyclic cover of a 3-manifold [4, 5, 7, 8]. We will use the notation of [7] throughout. For any nice space X and a homomorphism $\chi: H_1(X) \rightarrow \mathbb{Z}_m$, there is an induced m -fold cyclic cover $\tilde{X}$ of X together with a specific choice T of a generator for the group of covering transformations. Let $\tilde{H}_k(X, \mathbb{C})$ be the $e^{2\pi i/m}$ eigenspace for the action of T on $H_k(\tilde{X}, \mathbb{C})$ and $\tilde{\beta}(\tilde{X})$ its dimension. Now suppose M is a closed 3-manifold and $\chi: H_1(M) \rightarrow \mathbb{Z}_m$, then for some integer r we can find a 4-manifold V and a character $\chi': H_1(V) \rightarrow \mathbb{Z}_m$ such that $\partial(V, \chi') = r(M, \chi)$. Let $\sigma_1(\tilde{V})$ be the signature of the hermitian intersection pairing on $H_2(\tilde{V}, \mathbb{C})$ restricted to $\tilde{H}_2(\tilde{V})$. Define $\sigma(M, \chi) = 1/r(\sigma_1(\tilde{V}) - \text{Sign}(V))$ and $\eta(M, \chi) = \beta_1(\tilde{M})$.

§2. RATIONAL HOMOLOGY 3-SPHERES

THEOREM (2.1). *If M is a rational homology 3-sphere which embeds in S^4 then $H_1(M)$ can be written as $G_1 \oplus G_2$ and M has an almost framing $\mathcal{F}$ such that*

- (1) $G_1 \cong G_2$
- (2) If $\psi \in H^1(M, \mathbb{Z}_2)$ and $\psi(G_i) = 0$ for $i = 1$ or 2 , then $\mu(M, \psi\mathcal{F}) = 0$
- (3) $q_{\mathcal{F}}(G_i) = 0$ for $i = 1$ and 2
- (4) If $\chi: H_1(M) \rightarrow \mathbb{Z}_d$ where d is a power of a prime p and $\chi(G_i) = 0$ for $i = 1$ or 2 , then

$$|\sigma(M, \chi)| + |\rho - 1 - \eta(M, \chi)| \leq \rho$$

where $\rho = \frac{1}{2} \dim H_1(M, \mathbb{Z}_p)$.

Proof. M separates S^4 into two components, the closures of which we denote by V_1 and V_2 . Consider a point on M in S^4 , S^4 minus this point has a framing which we can isotope so that, along M -point, the first three vectors are tangent to M and the last is normal to M . This is because the inclusion $SO(3) \rightarrow SO(4)$ is a 2-equivalence. This gives us an almost framing $\mathcal{F}$ of M which extends to a framing on V_1 and V_2 . Let $H_1(V_1) = G_1$ and $H_1(V_2) = G_2$, then a Mayer–Vietoris sequence shows that $H_1(M) = G_1 \oplus G_2$. Moreover, under this isomorphism the map induced by inclusion $H_1(M) \rightarrow H_1(V_i)$ is the standard projection onto G_i .

(1) This is a result of Hantzsche [9]. The long exact sequence for the pair (S^4, V_1) shows $H_1(V_1) \cong H_2(S^4, V_1)$. By excision, $H_2(S^4, V_1) \cong H_2(V_2, M)$. Duality gives that $H_2(V_2, M) \cong H^2(V_2)$. Finally, by the universal coefficient theorem, $H^2(V_2) = H_1(V_2)$ (as both are torsion groups).

(2) Since the V_i are codimension zero submanifolds of S^4 , their intersection pairings are identically zero. If $\psi(G_2) = 0$ then ψ factors through $H_1(V_1) = G_1$.

Since V_1 has the homotopy type of a 3-complex, we have that $[V_1, SO(4)]$ maps onto $H^1(V_1, \pi_1(SO(4))) = H^1(V_1, \mathbb{Z}_2)$. Thus, we may change the extension of $\mathcal{F} \oplus \epsilon$ to V_1 by element of $[V_1, SO(4)]$ to obtain an extension of $\psi\mathcal{F} \oplus \epsilon$. Thus $\mu(M, \psi\mathcal{F}) = \text{Sign}(V_1) = 0$.

(3) We have G_i is the kernel of map induced by inclusion $H_1(M) \rightarrow H_1(V_j)$ where $i \neq j$. If we calculate $q_{\#}$ on G_1 using V_2 , then since the intersection pairing on V_2 is identically zero $q_{\#}(G_1) = 0$. See proposition 5.7 of [16]. Similarly, $q_{\#}(G_2) = 0$. It follows that $l(G_i \times G_j) = 0$ which is a result of Kawauchi and Kojima [14].

(4) If χ vanishes on G_2 , it extends to a character on $H_1(V_1)$. Hence the associated cover $\tilde{V}_1$ can be used to compute $\sigma(M, \chi)$. Below we will omit the subscript in V_1 .

The Mayer-Vietoris sequence shows that V is a rational ball and that $H_3(V, \mathbb{Z}_p) = 0$. By ([17], Proposition 1.1), $1 = e(V) = \sum (-1)^n \bar{\beta}_n(\tilde{V})$. We have $\bar{\beta}_3(\tilde{V}) = 0$ by ([17], Proposition 1.4) and $\bar{\beta}_0(\tilde{V}) = 0$ because $\chi \neq 0$. Therefore

$$\bar{\beta}_2(\tilde{V}) = 1 + \bar{\beta}_1(\tilde{V}).$$

By duality $\bar{H}_1(\tilde{V}, \tilde{M}) = \bar{H}_3(\tilde{V}) = 0$. Consider the exact sequence $\bar{H}_2(\tilde{V}) \rightarrow \bar{H}_2(\tilde{V}, \tilde{M}) \rightarrow \bar{H}_1(\tilde{M}) \rightarrow \bar{H}_1(\tilde{V}) \rightarrow 0$. The first map in the sequence can be given by a matrix for the intersection pairing on $\bar{H}_2(\tilde{V})$. Denote the nullity of this matrix by n . Then

$$|\sigma(M, \chi)| + n \leq \bar{\beta}_2(\tilde{V}).$$

By exactness we have

$$\eta(M, \chi) = n + \bar{\beta}_1(\tilde{V}).$$

Thus

$$(a) \quad |\sigma(M, \chi)| + \eta(M, \chi) \leq 2\bar{\beta}_1(\tilde{V}) + 1.$$

Finally we can obtain two upper bounds on $\beta_1(\tilde{V})$. One arises because the last map in the above exact sequence is onto, the second follows from ([17], Proposition 1.5).

$$(b) \quad \bar{\beta}_1(\tilde{V}) \leq \min \left\{ \frac{\eta(M, \chi)}{\rho - 1} \right\}.$$

The inequality in the statement of the theorem follows formally from (a) and (b). $\square$

ADDENDUM (2.2). Suppose that the map induced by the inclusion $\pi_1(M) \rightarrow \pi_1(V_1)$ is onto and χ vanishes on G_2 , then one has the following estimate on $\sigma(M, \chi)$ even if d is not a prime power.

$$|\sigma(M, \chi)| \leq \eta(M, \chi) + 1.$$

Proof. One has $\pi_1(\tilde{M}) \rightarrow \pi_1(\tilde{V})$ is also onto. Therefore, $\bar{H}_1(\tilde{V}, \tilde{M}) = 0$ and by duality $\bar{H}_3(\tilde{V}) = 0$. In this way one obtains (a) and the upper inequality in (b). $\square$

Remark. The inequality in (4) of (2.1) is always equality mod 2, whenever conditions (1) and (3) hold. This follows from the following more general result. Now let M be any closed 3-manifold and define a bilinear, symmetric form

$$\beta : H^1(M, \mathbb{Q}/\mathbb{Z}) \times H^1(M, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

by the formula $\beta(\chi, \chi) = -l(\hat{\chi}, \hat{\chi})$ where $\hat{\chi}$ is the image χ in $\text{Tor } H_1(M)$ under the map

$$H^1(M, \mathbb{Q}/\mathbb{Z}) \rightarrow [\text{Tor } H_1(M)]^* \xrightarrow[\approx]{(Adl)^{-1}} \text{Tor } H_1(M).$$

Now if $\chi : H_1(M) \rightarrow \mathbb{Z}_m$ we can view it as a map into $\mathbb{Q}/\mathbb{Z}$ by embedding $\mathbb{Z}_m$ in $\mathbb{Q}/\mathbb{Z}$

with the residue class of 1 mapping to $1/m$. Then we have the following equation in $\mathbb{Q}$

$$\sigma(M, \chi) + 2\bar{\beta}(\chi, \chi) = \eta(M, \chi) + \beta_l(M) + 1 \pmod{2} \quad (2.3)$$

where $\bar{\beta}(\chi, \chi)$ is any rational number which reduces to $\beta(\chi, \chi) \in \mathbb{Q}/\mathbb{Z}$. This can be proved using ([7], Theorem (3.6)). Now if conditions (1) and (3) of 3.1 hold, l is hyperbolic with respect to the splitting $G_1 \oplus G_2$. If χ vanishes on G_i , then $\beta(\chi, \chi) = 0$, and the inequality of (4) (2.1) holds mod 2.

§3. CONNECTED SUMS OF LENS SPACES

When does $L(m, q) \# (L(m', q'))$ embed in 4-space? Kawauchi and Kojima[14] considered this question. We provide our own proofs for (3.1) and (3.2) below as we feel it is simpler to avoid their classification of linking forms. Also we must prepare the ground in (3.2) for the proofs of (3.3) and (3.5). By Hantzsche ((1) of 2.1) m must equal m' . m must be odd as well since Epstein[6] showed this, if even a punctured $L(m, q)$ embeds in S^4 . This follows as well from (3.1) below, since $H_1(L(m, q)) = \mathbb{Z}_m$ has a generator g such that $l(g, g) = q/m$. This is well known, and can be seen geometrically from the description of $L(m, q)$ as $-m/q$ surgery to S^3 along the unknot. g is, in fact, given by the meridian of this unknot.

PROPOSITION (3.1). (Kawauchi and Kojima). *Let N be a (possibly open 3-manifold embedded in S^4 , then if $x \in H_1(N)$ is 2^k torsion, then $l(x, x)$ is 2^{k-1} torsion.*

Proof. Let γ be curve representing x , then γ and the null homology of $2^k\gamma$ must live in Q , some compact submanifold of N . Then $Q \times I$ embeds as a tubular neighborhood of Q . Let M denote the double of $Q = \partial(Q \times I)$. M embeds so $H_1(M) \approx G_1 \oplus G_2$ where $l(G_i \times G_i) = 0$. Let x' denote the image of x in $H_1(M)$ and write $x' = x_1 + x_2$ with $x_i \in G_i$. Then $l(x, x) = l(x', x') = l(x_1, x_1) + 2l(x_1, x_2) + l(x_2, x_2) = 2l(x_1, x_2)$. Since x is 2^k torsion, so are x_1 and $l(x_1, x_2)$. $\square$

PROPOSITION (3.2) (Kawauchi and Kojima). *If $L(m, q) \# -L(m, q')$ embeds in S^4 , then $L(m, q)$ and $L(m, q')$ are homotopy equivalent.*

Proof. We can write an element of $H_1(L(m, q) \# -L(m, q'))$ in terms of our preferred generators as (a, b) where a and b are integers determined mod m . The self linking of this element is $a^2q - b^2q'/m \in \mathbb{Q}/\mathbb{Z}$. We apply Theorem (2.1) with conclusions (1) and (3) but only need consider the linking form. G_1 is generated by an element (a, b) of order m and self linking zero. Therefore $a^2q = b^2q' \pmod{m}$. Suppose b and m share some prime factor, then by the above equation a has this same factor. This contradicts the fact that (a, b) has order m . Therefore, $q' = c^2q \pmod{m}$ where $c = a\bar{b}$ and $b\bar{b} = 1 \pmod{m}$. The homotopy classification of lens spaces now finishes the proof. Note that $(c, 1)$ also generates G_1 . $\square$

CONJECTURE. *If $L(m, q) \# -L(m, q')$ embeds in S^4 , then $L(m, q)$ and $L(m, q')$ are diffeomorphic.*

Zeeman[20] showed that $L(m, q)$ -point embeds in S^4 if m is odd. Thus, $L(m, q) \# -L(m, q)$ does embed if m is odd. We will prove the conjecture if m is a prime power or less than 231.

If $d \mid m$, let $\chi_d : H_1(L(m, q)) \rightarrow \mathbb{Z}_d$ denote the map which assigns the residue class of

1 to the element g . This is denoted simply as χ in Example (3.9) of [7]. An efficient formula for computing $\sigma(L(m, q), s\chi_d)$ (using continued fractions) is given there. We used this formula in making the computer calculations discussed below.

LEMMA (3.3). *If $L(m, q) \# -L(m, q')$ embeds in S^4 , then for some c such that $q' = c^2 q \bmod m$ and for every prime power divisor d of m , and for every s , $0 < s < d$*

$$\sigma(L(m, q), s\chi_d) = \sigma(L(m, q'), sc\chi_d).$$

Proof. We apply Theorem (2.1). By the Proof of (3.2), G_1 is generated by $(c, 1)$ where $q' = c^2 q \bmod m$. The homomorphisms $s(\chi_d \oplus -c\chi_d)$ vanish on G_1 . It is not hard to show that the η invariant of these covers of $L(m, q) \# -L(m, q')$ is 1. One only need picture the cover as a kind of ferris wheel. Now condition (4) of (2.1) yields the above equality. $\square$

THEOREM (3.4). *If $L(m, q) \# -L(m, q')$ embeds and m is a prime power or less than 231, then $L(m, q)$ and $L(m, q')$ are diffeomorphic.*

Proof. Suppose first that m is a prime power. There is a close relation [4, page 6] (or see ([7] p. 376)) between $\sigma(M, \chi)$ and the Atiyah and Singer α -invariant [3] for the action of the group of covering transformations on $\tilde{M}$. In fact, the equality of (3.3) leads to (and is equivalent to)

$$\alpha(S^3, T^s) = \alpha(S^3, T'^{cs})$$

for all $0 < s < m$ where $\bar{c}c = 1 \bmod m$ and T , respectively T' , acts on $S^3 \subset \mathbb{C}^2$ by sending (x, y) to $(wx, w^q y)$, respectively $(wx, w^{q'} y)$. Here w denotes $e^{2\pi i/m}$. The proof given by Atiyah and Bott ([2], Theorem 7.27) shows $\bar{c} = \pm 1 \bmod m$ or $\bar{c} = \pm q \bmod m$. In any of these cases $L(m, q)$ is diffeomorphic to $L(m, q')$.

Now if m is not a prime power, we can still apply (3.3). A computer search shows the Casson–Gordon invariants associated to characters of prime power order are sufficient to distinguish lens spaces with m odd and less than 231. $\square$

The space $L(231, 53)$ and $L(231, 86)$ are homotopy equivalent since $(23)^{253} = 86 \bmod 231$, but are not diffeomorphic. Moreover, the Casson–Gordon invariants for characters of prime power order appropriately matched are all equal, and their μ -invariants are equal. Thus using Theorem (2.1) we are unable to show that $L(231, 53) \# -L(231, 86)$ does not embed. These are the only pairs of lens spaces we know of with this property and the only such pair with $m = 231$. Addendum (2.2) applies and a computer check shows that the $L(231, 53) \# -L(231, 86)$ cannot embed in S^4 with the fundamental group surjecting onto the fundamental group of either component of the complement.

After we did the above work, we noticed the following connection with homology cobordism.

PROPOSITION (3.5). *If $L(m, q) \# -L(m, q')$ embeds in S^4 then $L(m, q)$ is integral homology cobordant to $L(m, q')$.*

Proof. We use the notation of the proof of (2.1). In (3.2) we showed that G_1 the kernel of $H_1(M) \rightarrow H_1(V_2)$ must be generated by $(c, 1)$. Therefore, $H_1(L(m, q) - pt)$ injects into $H_1(V_2)$. Since both groups are $\mathbb{Z}_m$, the inclusion is an isomorphism. The same goes for $L(m, q')$. Attaching a 3-handle to V_2 along the separating 2 sphere in $L(m, q) \# -L(m, q')$ creates a 4-manifold which is a homology cobordism. $\square$

Now we could prove have proved Theorem (3.4) by modifying the proof of the h -cobordism classification of lens spaces. By Smith theory ([7] 1.4 and 1.5) a prime power cover of an integral homology cobordism of lens spaces is a rational homology cobordism of lens spaces.

§4. SURGERY ON LINKS

We consider 3-manifolds obtained by surgery on the link illustrated in Fig. 1(a). J and K represent knots tied in components of the link and n is a positive integer indicating a number of full right-handed twists. Let M denote the manifold obtained by performing zero framed surgery along each component. A typical example where $n = 3$ is given in Fig. 1(b). Manifolds of this type when n is even present examples where stronger results are obtained by using the quadratic form (as opposed to the linking form). In working with framed link description of 3-manifolds we use the Kirby calculus[15]. We begin with some positive embedding results.

PROPOSITION (4.1). *If the knots J and K are slice knots, then M^3 embeds in S^4 .*

Proof. View S^3 as an equator of S^4 . Let J bound a disk D_1 in the upper hemisphere and K a disk D_2 in the southern hemisphere. Let $N(D_i)$ denote tubular neighborhoods of D_i in their hemispheres and $N(J)$ and $N(K)$ tubular neighborhoods of J and K in S^3 . Then $(S^3 \cup \partial N(D_1) \cup \partial N(D_2)) - N(J) - N(K)$ is M and is embedded in S^4 . $\square$

The converse to (4.1) is not true. Suppose K is unknotted. Beginning with the

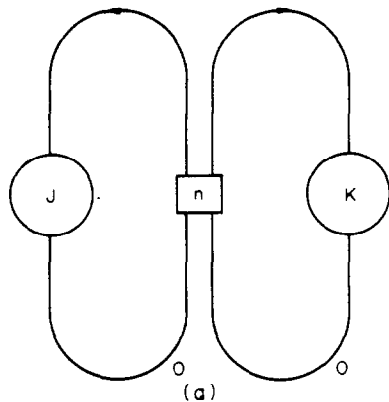


Fig. 1(a).

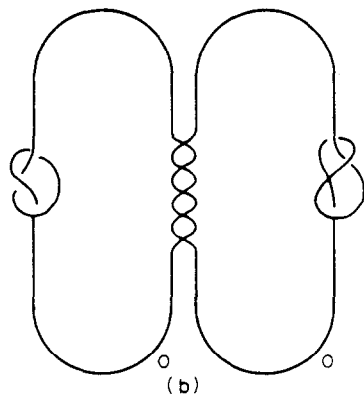


Fig. 1(b).

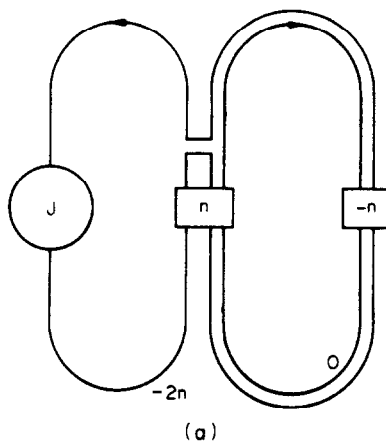


Fig. 2(a).

description given by Fig. 1(a), sliding J over K gives a description of M illustrated in Fig. 2(a). After isotopy this is given by Fig. 2(b). Figure 3(a) illustrates a second slide, and in 3(b) we draw the knotted component of the link in 3(a). If J is the knot illustrated in Fig. 3(c), then each component of the link in Fig. 3(a) is slice. The Proof of 4.1 shows that this manifold embeds. Of course, this is M where we have chosen K to be the unknot and J to be the knot in Fig. 3(c). J is not slice for $n \geq 1$ as for all $m > 1$, $\sigma_{[m/2]/m}(J) = 2$. See ([7], p. 363) for the definition of σ_q where $0 < q < 1$. By repeating this trick twice, one can find knots J and K with the above signatures both equal to 2 such that the resulting M embeds. The following theorem can be viewed as a weak converse to 4.1. We let μ denote the Arf invariant of a knot, taking values 0 or 1.

THEOREM (4.2). *Suppose M embeds in S^4 and let $n = 2^k m$ with m odd. We have*

$$\sigma_{s/d}(J) = \begin{cases} 0 \\ \text{or} \\ 2 \end{cases} \text{ and } \sigma_{s/d}(K) = \begin{cases} 0 \\ \text{or} \\ 2 \end{cases}$$

where s is an integer, $0 < s < d$ and d is a prime power dividing m or 2^{k-1} . If $k = 1$ or $k \geq 3$ then $\mu(J) = \mu(K) = 0$. If $k = 2$, then $\mu(J)$ or $\mu(K)$ must be zero. If $k = 2$,

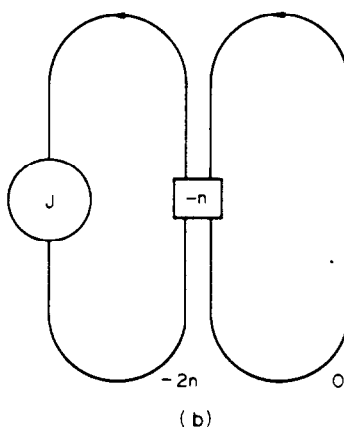


Fig. 2(b).

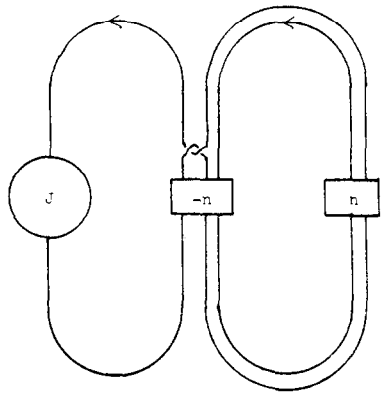


Fig. 3(a).

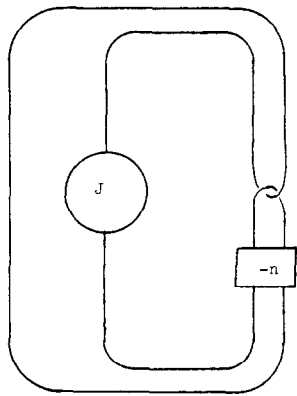


Fig. 3(b).

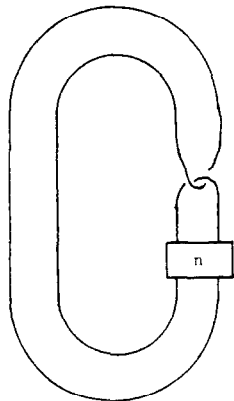


Fig. 3(c).

$\mu(J) = 0$ implies the above signature condition for J holds with $d = 4$. When $k = 1$, the signature condition must hold for J or K with $d = 2$.

Proof. Let x and y denote the positively oriented meridians of J and K respectively. Then $H_1(M) \approx \mathbb{Z}_n \oplus \mathbb{Z}_n$ with x and y corresponding to generators of the respective factors. A Seifert surface F for K intersects J in n points. Delete n small disks from F around each of these points and cap off F with a disk in the resewn

solid torus along K . This produces a surface representing a null-homology for $-nx$ and y intersects this surface in one point. Using this surface and a similar one for J one finds $l(x, x) = l(y, y) = 0$ and $l(x, y) = -1/n$.

We apply Theorem (2.1) and consider the possible splittings $H_1(M) = G_1 \oplus G_2$ satisfying conditions (1) and (3). Let $ax + by$ generate G_i , and d divide m or 2^{k-1} . $0 = l(ax + by, ax + by) = -2ab/n$. Therefore, $ab = 0 \pmod d$. Since $ax + by$ generates G_i , one of a or b must be invertible mod d and the other zero. It follows that G_1 is generated by an element equal to $x \pmod d$ and G_2 by an element equal to $y \pmod d$ (or vice versa).

If n is even, then $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and there are four isotopy classes of almost framings which we denote by $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$. The first entry is zero, respectively one, if the push off of x (using the rule in the definition of q) links x an even, respectively an odd, number of times. Similarly for the second entry. Then

$$q_{(i,j)}(x) = \frac{i}{2} \text{ and } q_{(i,j)}(y) = \frac{j}{2}.$$

According to Kaplan ([12], §4.2)

$$\begin{aligned} \mu_{(0,0)}(M) &= 0 & \mu_{(1,1)}(M) &= 8(\mu(J) + \mu(K)) + 2n \\ \mu_{(1,0)}(M) &= 8\mu(J) & \mu_{(0,1)}(M) &= 8\mu(K). \end{aligned}$$

Now the μ -invariant must vanish for at least three almost framings. Suppose $\mu(J) \neq 0$, then $\mu(K) = 0$ and $n = 4 \pmod 8$. This gives our conditions on the Arf invariants of J and K . If $k = 2$ and $\mu(J) = 0$, then an analysis using conditions 1, 2 and 3 of Theorem (2.1) shows that $\mathcal{F}$ is $(0, 0)$ or $(1, 0)$. One can then show that G_1 or G_2 must be generated by an element equal to $y \pmod 4$. Clearly if $k = 1$ then G_1 or G_2 must be generated by an element equal to z or $y \pmod 2$.

We now apply condition (4) of (2.1). If $\chi : H_1(M) \rightarrow \mathbb{Z}_d$ takes the value 1 on x and 0 on y (or vice versa), we have

$$|\sigma(M, s_\chi)| + |\eta(M, s_\chi)| \leq 1.$$

Assume $\chi(x) = 1$ and $\chi(y) = 0$. Slide J over K to arrive at the framed link L in Fig. 4 with indicated Seifert surface. Now χ takes the value 1 on the meridians of both components, so we can use ([7], Theorem (3.6)) to calculate the invariants. The Seifert matrix for L is formed from the Seifert matrix for J by taking the direct sum with the

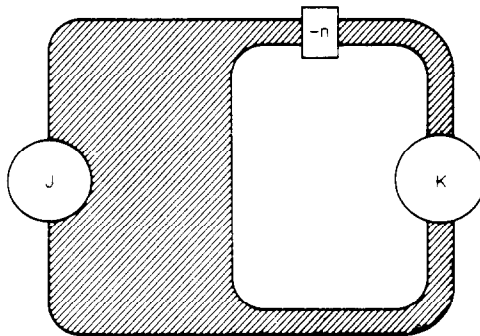


Fig. 4.

matrix $(-n)$. Therefore, $\sigma(M, s_\chi) = \sigma_{s/d}(J) - 1$ and $\eta(M, s_\chi) = 0$. Therefore, $\sigma_{s/d}(J)$ is zero or two (as these signatures are always even). A similar argument works for K . $\square$

Remark. Suppose now M is obtained by 0-framed surgery on a link of two components with linking number n . Let J and K denote the components viewed as knots in S^3 . Proposition (4.1) and Theorem (4.2) (if we replace 2 by ± 2 in the signature conditions) hold by essentially the same arguments. One estimates $\sigma(M, \chi)$ and $\eta(M, \chi)$ in this case by comparing M to the manifold obtained by 0-framed surgery along J or K .

§5. DOUBLE NULL-CONCORDANCE OF KNOTS

We were initially led to our examination of the embedding problem when trying to apply the techniques of Casson and Gordon to the problem of double null-concordance of knots, as described below. We would like to thank Danny Ruberman for his participation and contributions during those early discussions. We should also mention that Ruberman has recently shown that the Casson–Gordon invariant can be applied in the study of doubly null-concordant knots in higher dimensions.

A knot in S^3 is said to be *doubly null-concordant* if it is the intersection of a standardly embedded S^3 in S^4 with an unknotted 2-sphere in S^4 . In [18] it is proved that for a knot to be doubly null-concordant its Seifert pairing, determined from a surface F , must vanish on two submodules of $H_1(F)$ having equal rank and whose direct sum is $H_1(F)$. We call a knot with such a pairing algebraically doubly null-concordant.

Using the methods of Section 4, we will produce examples of slice knots which are algebraically doubly null-concordant but which are not doubly null-concordant.

Example. The knot L indicated in Fig. 5 is algebraically doubly null-concordant and is slice, but is not doubly null-concordant.

Reason. L has Seifert form

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

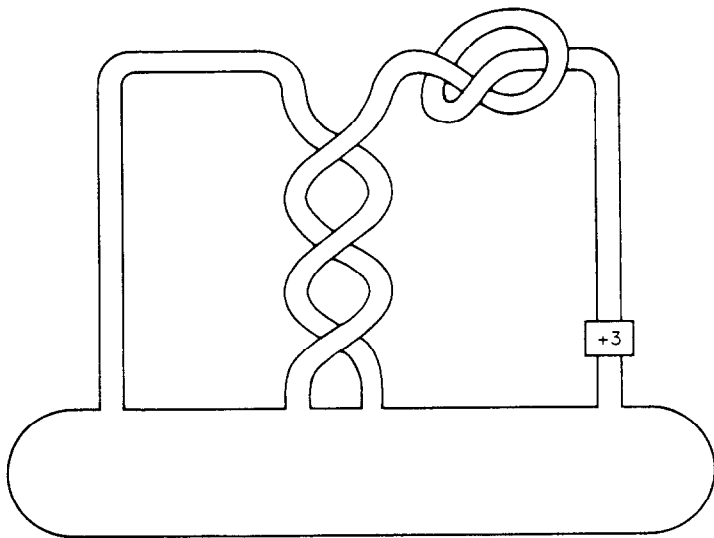


Fig. 5.

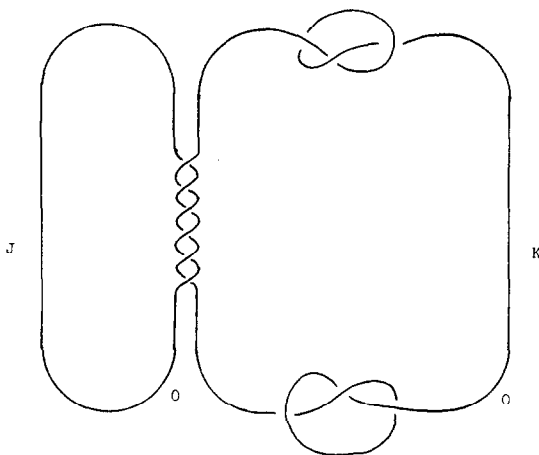


Fig. 6.

and is hence clearly algebraically doubly null-concordant. As one of the bands on the Seifert surface is untwisted and unknotted, L is slice.

If L were doubly null-concordant the double cover of S^3 branched along L would embed in S^4 . This is because the 2-fold cover of S^4 branched along an unknotted 2-sphere is again S^4 . Using the techniques of [1] the double cover of S^3 branched along L has a framed link description as illustrated in Fig. 6. As $\sigma_{1/3}(K) = 4$, Theorem 5.2 applies to show this manifold does not embed in S^4 .

Remark. A genus 1 knot which is algebraically doubly null-concordant bounds a Seifert surface which is built from a disk by adding two untwisted bands. An analysis as done above and in Section 5 can be used to show that if the knot is doubly null-concordant, then $\sigma_{s/d}(J_1) = \sigma_{s/d}(J_2) = 0$, where d is a prime power which divides the square root of the determinant of the knot. Here J_1 and J_2 denotes the knots into which the bands are tied.

§6. HOMOLOGY $S^1 \times S^2$'s

In this section, we relate the embedding problem for manifolds with the integral homology of $S^1 \times S^2$ to the cobordism problem for knots. Let M be such a manifold, if we perform honest surgery to a curve representing a generator for $H_1(M)$, one obtains a homology sphere. Then M can be obtained by doing 0-framed surgery along a knot K in some (in general many) homology 3-sphere S .

If K is a knot in a homology sphere S , let M_K denote 0-framed surgery to S along K .

PROPOSITION 6.1. M_K embeds in a homology 4-sphere if and only if K is slice in some homology 4-ball D .

Proof. If K is slice in D then M_K embeds in the double of D by a construction similar to that given in the proof of (4.1). On the other hand, if $S^1 \times S^2$ embeds in a homology sphere then one component of the complement, say R , will be a homology circle. Now attach a 2-handle with zero framing to $S \times I$ along K . The boundary is S union M_K . Attach R to the result along M_K to obtain D . The core of the 2-handle is a slice disk for K . $\square$

Thus if M_K embeds K must be algebraically slice. This is also a corollary of Kawauchi's work [13]. The work of Casson and Gordon [4, 5, 8] gives further obstructions to knots being slice in homology 4-balls. We remark that all known invariants that can be used to show a knot K is not slice can in fact be viewed as invariants of M_K .

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*Institute for Advanced Study, Princeton, and
Louisiana State University, Baton Rouge
Indiana University, Bloomington.*